Eigenvalue and Eigenvector Derivatives of Nonlinear Eigenproblems

Zhong-sheng Liu* and Su-huan Chen†
Jilin University of Technology, Changchun, 130022, People's Republic of China

Introduction

THE derivatives of eigenvalues and eigenvectors with respect to design parameters are useful, if not essential, for design sensitivity and structural optimization studies.

The computational problem of the derivatives of eigenvalues and eigenvectors of linear eigenvalue problems has been treated adequately in the past.[1-4] However, recent work on the nonlinear case has shown that the computational problem of the eigenvalue and eigenvector derivatives of nonlinear eigenproblems with respect to design parameters remains a dilemma to be studied. Though this problem has been addressed in Ref. 5 and the six theorems of the derivatives of eigenvalues and eigenvectors of nonlinear eigenproblems have been derived, the first four theorems are not correct and the assumptions stated there are not often satisfied, as pointed out in Ref. 6.

In this Note, the explicit formulas for the first-order derivatives of distinct eigenvalues are derived for self-adjoint nonlinear eigenproblems and the three computational methods for the derivatives of eigenvectors proposed.

Contents

The nonlinear eigenvalue problem discussed here is

$$([K] - \lambda[M])\,\{X\} = \{0\} \tag{1a}$$

$$\{X\}^T[M]\{X\} = 1 \tag{1b}$$

where

$$[K] \equiv [K(p,\lambda)], \qquad [M] \equiv [M(p,\lambda)]$$

$[K]$ and $[M]$ are $n \times n$ symmetric matrices representing nonlinear operators on the unrepeated eigenvalue λ; $\{X\}$ is the corresponding eigenvector normalized by matrix $[M]$; the superscript T denotes transpose of; and p is a design parameter of concern. It is assumed that λ and $\{X\}$ are differentiable functions of p. The elements of $[K]$ and $[M]$, k_{ij} and m_{ij} ($i, j = 1, \ldots, N$), must also be differentiable functions of p and λ and are, in general, nonlinear functions of λ and p.

The differentiation of Eq. (1) with respect to the design parameter p gives

$$([K] - \lambda[M])\,\{X\}' = \{F\} \tag{2a}$$

$$\{X\}^T[M]\{X\}' = b \tag{2b}$$

where

$$b = -\{X\}^T([M]'_p + \lambda'[M]'_\lambda)\,\{X\}/2 \tag{3a}$$

$$\{F\} = (\lambda([M]'_p + \lambda'[M]'_\lambda) + \lambda'[M] - [K]'_p - \lambda'[K]'_\lambda)\{X\} \tag{3b}$$

$$\lambda' \equiv d\lambda/dp, \qquad X' \equiv d\{X\}/dp, \qquad [M]'_p \equiv \partial[M]/\partial p$$

$$[M]'_\lambda \equiv \partial[M]/\partial\lambda, \qquad [K]'_p \equiv \partial[M]/\partial p$$

$$[K]'_\lambda \equiv \partial[K]/\partial\lambda$$

Equation (2) forms a set of linear algebraic equations with $\{X\}'$ and λ' unknown.

Since λ is a distinct eigenvalue, matrix $([K] - \lambda[M])$ is singular and rank $([K] - \lambda[M])$ is $(n-1)$. Based on the theory about a set of linear equations, the condition that a solution $\{X\}'$ exists in Eq. (2a) can be stated as

$$\{X\}^T\,\{F\} = 0 \tag{4}$$

From Eqs. (3b) and (4), we can get

$$\lambda' = \frac{\{X\}^T\,([K]'_p - \lambda[M]'_p)\,\{X\}}{1 + \{X\}^T\,(\lambda\,[M]'_\lambda - [K]'_\lambda)\,\{X\}} \tag{5}$$

To determine $\{X\}'$ from Eq. (2), we can use the following three methods:

Method I

Let

$$\{X\}' = [V_1]\,\{\alpha_l\} + \{V_2\}\alpha_2 \tag{6}$$

where $\{\alpha_1\} \in (n-1)$ − vector and $\alpha_2 \in$ scalar coefficient that are to be determined; $\{V_2\} \in n$ − vector and $[V_1] \in n \times (n-1)$ matrix given by the singular-value decomposition (SVD)[7] of $([K] - \lambda[M])$

$$([K] - \lambda\,[M]) = [[V_1]\;\vdots\;\{V_2\}]\begin{bmatrix}\Sigma_1 & \\ & 0\end{bmatrix}\begin{bmatrix}[V_1]^T \\ \{V_2\}^T\end{bmatrix}$$

$$= [V_1]\,\Sigma_1\,[V_1]^T \tag{7a}$$

where

$$[V_1]^T\,[V_1] = [I] \tag{7b}$$

$$\{V_2\}^T\,\{V_2\} = 1 \tag{7c}$$

$$[V_1]^T\,\{V_2\} = \{0\} \tag{7d}$$

and Σ_1 is a diagonal matrix with nonzero singular values on the diagonals.

Substituting Eqs. (7a) and (6) into Eq. (2a) and noting that $[V_1]^T\,\{V_2\} = \{0\}$, one has

$$\{\alpha_1\} = \Sigma_1^{-1}[V_1]^T\,\{F\} \tag{8a}$$

Substituting Eq. (6) into Eq. (2b) and noting that $\{X\}^T[M]\,\{V_2\} = 0$, one has

$$\alpha_2 = b/(\{X\}^T[M]\{V_2\}) \tag{8b}$$

Therefore, one obtains

$$\{X\}' = [V_1]\Sigma_1^{-1}\,[V_1]^T\{F\} + \{V_2\}[b/(\{X\}^T[M]\{V_2\})] \tag{9}$$

Method II

Now we try to combine Eqs. (2a) and (2b) into one nonsingular linear equation system with N equations.

Multiplying Eq. (2b) by $\mu\,[M]\{X\}$ yields

$$\mu\,[M]\{X\}\{X\}^T[M]\{X\}' = b\,\mu[M]\{X\} \tag{10}$$

Received Jan. 24, 1992; revision received June 2, 1992; accepted for publication July 10, 1992.

*Assistant Professor, Department of Mechanics; currently Associate Professor, Department of Mechanics, Peking University, Beijing 100871, People's Republic of China.
†Professor, Department of Mechanics.

where μ is a nonzero scalar parameter. Adding the two sides of Eq. (10) to those of Eq. (2a) gives

$$[A]\{X\}' = \{F\} + b\,\mu\,[M]\{X\} \tag{11}$$

where

$$[A] \equiv [K] - \lambda[M] + \mu[M]\{X\}\{X\}^T[M] \tag{12}$$

It can be shown that A is nonsingular if $\mu \neq 0$. Thus, $\{X\}'$ can be solved from Eq. (11) using standard techniques

$$\{X\}' = [A]^{-1}(\{F\} + b\,\mu[M]\{X\}) \tag{13}$$

where μ should be given from the standpoint of the condition number of the matrix A.

Method III

One of the more popular solution techniques for derivatives of eigenvectors of linear eigenproblem is due to Nelson.[3] Here we extend the Nelson method to incorporate the nonlinear eigenproblem.

We temporarily replace the normalization condition, Eq. (1b), by the requirement that the largest component $\bar{x}_m$ of the eigenvector under consideration be equal to 1. Denoting this renormalized vector $\{\bar{X}\}$ and assuming that its largest component is the mth one, we replace Eq. (1b) by

$$\bar{x}_m = 1 \tag{14}$$

and Eq. (2b) by

$$\mathrm{d}\bar{x}_m/\mathrm{d}p = 0 \tag{15}$$

Equation (2a) is valid with $\{X\}$ replaced by $\{\bar{X}\}$, but Eq. (15) is used to reduce its order by deleting the mth row and mth column. When the eigenvalue λ is distinct, the reduced system is not singular and may be solved by standard techniques.

To retrieve the derivative of the eigenvector with the original normalization of Eq. (1b), we note that

$$\{X\} = x_m\{\bar{X}\} \tag{16}$$

so that

$$\{X\}' = (\mathrm{d}x_m/\mathrm{d}p)\ \{\bar{X}\} + x_m\{\bar{X}\}' \tag{17}$$

and $\mathrm{d}x_m/\mathrm{d}p$ can be obtained by substituting Eq. (17) into Eq. (2b) to obtain

$$\mathrm{d}x_m/\mathrm{d}p = x_m b - x_m^2\{X\}^T[M]\{\bar{X}\}' \tag{18}$$

The methods presented here apply to many problems in dynamics and other branches of applied mathematics, such as the theories of oscillation, elasticity, optimization, and a dynamic finite-element system[8] with frequency-dependent mass and stiffness matrices.

Consider an example of a nonlinear eigenproblem

$$[K]\{X\} = \lambda[M]\{X\}$$

with the eigenvalue-dependent mass and stiffness matrices given by

$$[M] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 40 & 10 \\ 0 & 10 & 20 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1.75 \\ 0 & 1.75 & 2 \end{bmatrix}$$

$$[K] = p\begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix} + \lambda^2 \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 2 & 0.865 \\ 0 & 0.865 & 1 \end{bmatrix}$$

where λ and p are, respectively, the eigenvalue and design parameter under consideration.

The three positive eigenvalues λ and associated eigenvector $\{X\}$ at $p = 1.0$ are

$$\begin{aligned} &\lambda_1 = 1.019760, && \lambda_2 = 4.42136, && \lambda_3 = 10.159744 \\ &X_1 = \{0.0000000, && -0.0907256, && -0.1283053\}^T \\ &X_2 = \{0.1880301, && 0.0000000, && 0.0000000\}^T \\ &X_3 = \{0.0000000, && 0.1099595, && -0.1555062\}^T \end{aligned}$$

Using Eq. (4), we can give the derivatives of the three eigenvalues with respect to design parameter p (at $p = 1.0$) as follows: $\mathrm{d}\lambda_1/\mathrm{d}p = 0.964388$, $\mathrm{d}\lambda_2/\mathrm{d}p = 3.535534$, and $\mathrm{d}\lambda_3/\mathrm{d}p = 8.256314$. The derivatives of eigenvectors can also be obtained by these same three methods.

Conclusions

This Note addressed the computation of the derivatives of eigensolutions in a nonlinear eigenproblem with respect to design parameters. The formulas for eigenvalue derivatives are presented and the three methods for eigenvector derivatives also proposed: the first is the SVD method, the second the direct method based on a transformation of Eqs. (2a) and (2b) into a nonsingular linear equation system that can be solved using standard algorithms for linear equation systems, and the last approach an extension of the Nelson method. The present three methods apply to distinct eigenvalues. In the case of repeated eigenvalues and closely spaced eigenvalues, the SVD method can keep track of the conditioning of the matrix A, so it is able to give an indication of the distinctness of the eigenvalue under consideration. This is of importance in any practical implementation. Further work is needed to extend the computational algorithm to include the case of repeated or closely spaced eigenvalues.

Acknowledgment

The study was supported by the National Natural Science Foundation of China and Open Laboratory of CAD/CAM Technology for Advanced Manufacturing, Academia Sinica, Shenyang City.

References

[1]Fox, R. L., and Kapoor, M. P., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal,* Vol. 6, No. 12, 1968, pp. 2426–2429.

[2]Haftka, R. T., and Adelmen, R. H., "Sensitivity Analysis of Discrete Structural Systems," *AIAA Journal,* Vol. 24, No. 5, 1968, pp. 823–832.

[3]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal,* Vol. 14, No. 9, 1976, pp. 1201–1205.

[4]Chen, S. H., and Pan, H. H., "Design Sensitivity Analysis of Vibration Modes by Finite-Element Perturbation," *Proceedings of the 4th IMAC* (Los Angeles, CA), Feb. 1986, pp. 38–43.

[5]Jankovic, M. S., "Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem," *AIAA Journal,* Vol. 26, No. 2, 1988, pp. 204, 205.

[6]Andrew, A. L., and Chu, K. W. E., "Comment on Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem," *AIAA Journal,* Vol. 29, No. 7, 1991, pp. 1182–1184.

[7]Golub, G. H., and Vanloan, C. E., *Matrix Computations,* 2nd ed., Johns Hopkins Univ. Press, Baltimore, MD, 1990.

[8]Przemieniecki, J. S., *Theory of Matrix Structural Analysis,* McGraw-Hill, New York, 1968, pp. 281–287.